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Some properties of inverses of the full matrices

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ABSTRACT

In this paper, we discuss the problem of the number of elements equal to zero included in the inverse matrix, in case when the given matrix is full or it satisfies some specific algebraic-geometrical conditions.

1. Introduction

Origins of this paper lie in our interests in the, so called, full matrices [1].

Definition 1. Matrix $A = [a_{ij}] \in \mathbb{M}_{n\times m}(\mathbb{C})$ is said to be a full matrix, if all its elements are different from zero ($a_{ij} \neq 0$).

As a result of the research made by the way of preparing the paper [2], there arose a need for testing some numerical algorithms. Using the full matrices of the known in advance eigenvalues then seemed to be the most promising idea. The problem of generating such kind of matrices was one of the main reasons for preparing the paper [3]. Moreover, investigations of some other properties of the full matrices also seemed to be interesting. One of the questions, which appeared at that time, concerned the possible number of elements equal to zero included in the inverse to the given full matrix. In this paper, we answer this question, as well as many other related questions. For example, we note that some completely natural properties of the algebraical-geometric nature of the considered matrix have an influence on the number of zeros in its inverse (see Theorems 2 and 4 and Remark 1).

Subjects of this matter were extremely inspiring for us, however for sure, they do not determine all the possible problems. Let us note that the full matrices (or the "almost" full matrices) are known in pure mathematics [4] and sciences. They appear, among others, in the finite element method, computer graphics, data compression, filtration and optics. Special matrices, like the full matrices discussed here, have very wide applications in system identification, parameter estimation and signal processing, including the transformation between companion matrices [5], Sylvester matrix equations [6,7], general (coupled) matrix equations [8–12] and their solutions. These matrix equations can be solved by using the iterative algorithms based on the least squares and/or gradient search. For example, the following Refs. [6–12] present the novel gradient-based iterative (GI) method [6,8–12] and least squares based iterative methods [7,9,11,12] with highly computational efficiencies for solving (coupled) matrix equations and good stability performances, based on the hierarchical identification principle [9].

We begin the discussion with the analysis of the few special cases.

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2. Case of 2 × 2 matrices

Certainly, if matrix $A \in GL(2, \mathbb{R})$ is a full matrix, then matrix A^{-1} is also a full matrix since the following relation holds true

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

3. Case of 3 × 3 matrices

We prove the following theorem.

Theorem 1. *Let $A \in GL(3, \mathbb{R})$. If A is a full matrix then its inverse includes at most three zeros (in addition, none two of them can lie in the same row or in the same column either).*

Proof. Suppose, contrary to our claim, that in matrix A^{-1} there exist two zeros in one column. For instance, let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \quad \text{and} \quad (\det A)A^{-1} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & 0 & 0 \end{bmatrix}^T.$$

Then we have

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = 0$$

which, in view of the assumption about A being a full matrix, means that there exist numbers $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{bmatrix} b \\ e \end{bmatrix} = \alpha \begin{bmatrix} a \\ d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ f \end{bmatrix} = \beta \begin{bmatrix} a \\ d \end{bmatrix}.$$

Thus

$$A = \begin{bmatrix} a & \alpha\, a & \beta\, a \\ d & \alpha\, d & \beta\, d \\ g & h & k \end{bmatrix}$$

and

$$\det A = ad \begin{vmatrix} 1 & \alpha & \beta \\ 1 & \alpha & \beta \\ g & h & k \end{vmatrix} = 0,$$

which is impossible. This completes the proof. □

From the proof of Theorem 1 the following more general fact can be obtained.

Remark 1. Let $\mathcal{N}_3$ be the subset of $GL(3, \mathbb{R})$ of all matrices containing at most one zero in every row and in every column. Then we have

$$\mathcal{N}_3^{-1} = \mathcal{N}_3,$$

where $\mathcal{N}_3^{-1} = \{A^{-1} : A \in \mathcal{N}_3\}$.

The following examples consolidate the thesis of Theorem 1. Let us consider the full matrices with the inverse matrices possessing three, two, one and no zeros, respectively (hereafter, we will denote as I_n the $n \times n$ unit matrix, for every $n \in \mathbb{N}$):

$$\begin{bmatrix} 1 & 1 & 6 \\ 1 & 1 & 4 \\ 2 & 3 & 12 \end{bmatrix} \begin{bmatrix} 0 & 6 & -2 \\ -4 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} = 2I_3,$$

$$\begin{bmatrix} 1 & 1 & 6 \\ 1 & 1 & 7 \\ 2 & 3 & 12 \end{bmatrix} \begin{bmatrix} -9 & 6 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} = -I_3,$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 1 & 7 \\ 2 & 3 & 12 \end{bmatrix} \begin{bmatrix} -9 & 3 & 2 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix} = -2I_3,$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 3 & 7 \\ 2 & 3 & 12 \end{bmatrix} \begin{bmatrix} 15 & 3 & -8 \\ 2 & 2 & -2 \\ -3 & -1 & 2 \end{bmatrix} = 2I_3.$$

4. Case of 4 × 4 matrices

We start with the theorem which corresponds with Theorem 1 for the 4×4 matrices, but it concerns not only the full matrices.

Theorem 2. *Let $A \in GL(4, \mathbb{R})$. If no 3×3 submatrix B of A has two proportional (linear dependent) columns or rows, then A^{-1} contains at most four zeros (in addition, none two of them can lie in the same row or in the same column either).*

Proof. Let

$$A = \begin{bmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}.$$

On the contrary, suppose that a certain column of matrix A^{-1} contains two zeros. Without loss of generality, we can take that

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b & c & d \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} = 0.$$

In view of the theorem assumptions, it means that there exist multipliers $\alpha, \beta, \alpha_1, \beta_1 \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{bmatrix} a \\ a_1 \\ a_2 \end{bmatrix} = \alpha \begin{bmatrix} b \\ b_1 \\ b_2 \end{bmatrix} + \beta \begin{bmatrix} c \\ c_1 \\ c_2 \end{bmatrix}$$

and

$$\begin{bmatrix} d \\ d_1 \\ d_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} b \\ b_1 \\ b_2 \end{bmatrix} + \beta_1 \begin{bmatrix} c \\ c_1 \\ c_2 \end{bmatrix}.$$

Hence

$$-\beta_1 \begin{bmatrix} a \\ a_1 \\ a_2 \end{bmatrix} + \left(\alpha\beta_1 - \alpha_1\beta\right) \begin{bmatrix} b \\ b_1 \\ b_2 \end{bmatrix} + \beta \begin{bmatrix} d \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$-\alpha_1 \begin{bmatrix} a \\ a_1 \\ a_2 \end{bmatrix} + \left(\alpha_1\beta - \alpha\beta_1\right) \begin{bmatrix} c \\ c_1 \\ c_2 \end{bmatrix} + \alpha \begin{bmatrix} d \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

that is

$$\begin{vmatrix} a & b & d \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix} = \begin{vmatrix} a & c & d \\ a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \end{vmatrix} = 0,$$

which designates that all elements in some column of matrix A^{-1} are equal to zero, which is impossible. Obtained contradiction ends the proof of theorem. □

The following example of full matrix illustrates the above theorem:

$$\begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = 3I_4.$$

Next example shows that inverse of the full matrix $A \in GL(4, \mathbb{R})$ can possess exactly four zeros and does not have to satisfy the assumptions of Theorem 2:

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} = I_4.$$

More generally, if

$$A(a,b)=\begin{bmatrix}-2 & 1 & 1 & 1\\ 1 & -2 & 1 & 1\\ 1 & 1 & a & 1\\ 1 & 1 & 1 & b\end{bmatrix} \quad \text{and} \quad a(b+2)\neq 5-2b,$$

then

$$\tilde{A}(a,b)=3\big(a(b+2)+2b-5\big)A(a,b)^{-1}=\begin{bmatrix}4-b-a(1+2b) & (a-1)(1-b) & 3(b-1) & 3(a-1)\\ (a-1)(1-b) & 4-b-a(1+2b) & 3(b-1) & 3(a-1)\\ 3(b-1) & 3(b-1) & 3(b+2) & -9\\ 3(a-1) & 3(a-1) & -9 & 3(a+2)\end{bmatrix}.$$

Hence

$$\tilde{A}(-2,b)=3\begin{bmatrix}b+2 & b-1 & b-1 & -3\\ b-1 & b+2 & b-1 & -3\\ b-1 & b-1 & b+2 & -3\\ -3 & -3 & -3 & 0\end{bmatrix},$$

$$\tilde{A}(-2,1)=-9\begin{bmatrix}-1 & 0 & 0 & 1\\ 0 & -1 & 0 & 1\\ 0 & 0 & -1 & 1\\ 1 & 1 & 1 & 0\end{bmatrix}$$

and

$$a(1+2b)=4-b \Longrightarrow \tilde{A}(a,b)=3\begin{bmatrix}0 & \frac{1}{3}(a-1)(1-b) & b-1 & a-1\\ \frac{1}{3}(a-1)(1-b) & 0 & b-1 & a-1\\ b-1 & b-1 & b+2 & -3\\ a-1 & a-1 & -3 & a+2\end{bmatrix}$$

and, finally, if $a, b \notin \{-2, 1\}$, $a\,(1+2b)\neq 4-b$ then $\tilde{A}(a,b)$ is the full matrix.

Considering the above examples we certainly still do not have any answer to our fundamental question: how many zeros can the inverse of the full matrix $A \in GL(4,\mathbb{R})$ possess? The answer is contained in the theorem given below.

Theorem 3. *Inverse of a full matrix $A \in GL(4,\mathbb{R})$ possesses at most seven elements equal to zero (in addition, in each row and in each column we have at most two zero elements).*

Proof. Let $A \in GL(4,\mathbb{R})$ be a full matrix. We need to prove that

(i) if $\det A \neq 0$, then A^{-1} includes in each row and in each column at most two zeros,
(ii) if $\det A \neq 0$, then A^{-1} does not possess a zero 2×2 submatrix.

First we prove (i). Let

$$A=\begin{bmatrix}a & b & c & d\\ a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\end{bmatrix} \quad \text{and} \quad A^{-1}=\begin{bmatrix}\alpha & \beta & \gamma & \delta\\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3\end{bmatrix}.$$

Conversely, suppose that $\alpha_3=\beta_3=\gamma_3=0$, that is

$$\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=\begin{vmatrix}a & b & c\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=\begin{vmatrix}a & b & c\\ a_1 & b_1 & c_1\\ a_3 & b_3 & c_3\end{vmatrix}=0.$$

We show that we have then also

$$\begin{vmatrix}a & b & c\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=0. \tag{1}$$

If the row $\begin{bmatrix}a_3 & b_3 & c_3\end{bmatrix}$ would be proportional, respectively, to the row $\begin{bmatrix}a & b & c\end{bmatrix}$ or $\begin{bmatrix}a_1 & b_1 & c_1\end{bmatrix}$ or $\begin{bmatrix}a_2 & b_2 & c_2\end{bmatrix}$, then such $\xi \in \mathbb{R}$ would exist that we would have, respectively

$$\begin{vmatrix}a & b & c\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=\xi\begin{vmatrix}a_3 & b_3 & c_3\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=0$$

or

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \xi \begin{vmatrix} a & b & c \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \xi \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

which means that $\delta_3 = 0$, contrary to the invertibility of matrix A.

If the row $\begin{bmatrix} a_3 & b_3 & c_3 \end{bmatrix}$ would not be linearly dependent on any of the rows $\begin{bmatrix} a & b & c \end{bmatrix}$, $\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}$, $\begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}$, respectively, then

$$\begin{aligned} \begin{bmatrix} a_3 & b_3 & c_3 \end{bmatrix} &= \mu \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} + \delta \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \\ &= \mu_1 \begin{bmatrix} a & b & c \end{bmatrix} + \delta_1 \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \\ &= \mu_2 \begin{bmatrix} a & b & c \end{bmatrix} + \delta_2 \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}, \end{aligned}$$

where $\mu, \mu_1, \mu_2, \delta, \delta_1, \delta_2 \in \mathbb{R} \setminus \{0\}$. From this we get

$$\mu_1 \begin{bmatrix} a & b & c \end{bmatrix} + (\delta_1 - \delta) \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} - \mu \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} = 0$$

which also implies equality (1) and, again, is impossible.

Now we prove (ii). Let us suppose that A^{-1} includes 2×2 zero submatrix. By using the notation as above, without violating the generality of considerations, we take the assumption

$$\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 0$$

which means that, for example, the rows $\begin{bmatrix} \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}$and $\begin{bmatrix} \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$are linearly dependent and, for example

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0,$$

and, in consequence, (we have $A = (A^{-1})^{-1}$) we obtain $a_3 = 0$ which is in contradiction with the fact that A is a full matrix. $\square$

Now we present the examples of 4×4 full matrices, inverses of which possess the number of zeros equal to $0, 1, \ldots, 7$, respectively. Let

$$A(a) = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a \\ 2 & 3 & 2 & 3 \end{bmatrix}, \qquad B(a) = \begin{bmatrix} 1 & -2 & 2 & -1 \\ -2 & 4 & -5 & 2 \\ 1 & -2 & 4 & -2 \\ -1 & 1 & -2 & a \end{bmatrix},$$

and

$$C(a) = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & a \end{bmatrix}, \quad a \in \mathbb{C}$$

and let $\mathrm{Z}(M)$ denotes the number of zero elements in matrix M. Then we have

$$\mathrm{Z}\big(A(a)^{-1}\big) = \begin{cases} 6 & \text{for } a = \frac{3}{2}, \\ 5 & \text{for } a = 3, \\ 4 & \text{for } a \notin \left\{1, \frac{3}{2}, 3\right\}, \end{cases}$$

$$\mathrm{Z}\big(B(a)^{-1}\big) = \begin{cases} 7 & \text{for } a = 1, \\ 4 & \text{for } a \in \left\{\frac{1}{2}, \frac{2}{3}\right\}, \\ 3 & \text{for } a \notin \left\{\frac{1}{2}, \frac{2}{3}, 1\right\}, \end{cases}$$

$$Z\big(C(a)^{-1}\big) = \begin{cases} 3 & \text{for } a = 1, \\ 2 & \text{for } a = 2, \\ 1 & \text{for } a \in \left\{-\dfrac{1}{2}, \dfrac{4}{3}\right\}, \\ 0 & \text{for } a \notin \left\{-\dfrac{1}{2}, \dfrac{1}{3}, 1, \dfrac{4}{3}, 2\right\}. \end{cases}$$

5. Generalizations

In this section we present two theorems generalizing Theorems 1–3. First of them will be prefaced by introducing the auxiliary idea.

Definition 2. We say that a system of vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ is properly linear dependent if there exist the nonzero multipliers $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \mathbf{0},$$

where $\mathbf{0}$ denotes the zero vector of $\mathbb{R}^n$.

Theorem 4. *Let $A \in GL(n, \mathbb{R})$ and $k \in \{1, 2, \ldots, n\}$. Let B denotes a submatrix of matrix A obtained by removing the k-th row from the matrix A. Suppose that for any $(n-1) \times (n-1)$ submatrix C of B no set of $n-2$ rows of C is a properly linear dependent system. Then the k-th column of the inverse matrix of A possesses at most one zero.*

Proof. Let us suppose, on the contrary, that the k-th column of A^{-1} possesses at least two zeros. Then two different $(n-1) \times (n-1)$ submatrices C_1 and C_2 of B are singular. In view of the assumption it means that the sets of all rows of C_1 and C_2, respectively, are the properly linear dependent systems. Hence, it easily results that any one set of $n-1$ rows of the matrix B is the properly linear dependent system, that is, each $(n-1) \times (n-1)$ submatrix of B is singular. So, the k-th column of A^{-1} would be the zero vector which is impossible. □

Corollary 1. *Let $A \in GL(n, \mathbb{R})$. If the assumption of Theorem 4 holds for every $k \in \{1, 2, \ldots, n\}$, then every column of A^{-1} possesses at most one zero.*

Moreover, if the assumption of Theorem 4 holds for every $k \in \{1, 2, \ldots, n\}$ and for both matrices A and A^T simultaneously, then every row and every column of A^{-1} possesses at most one zero.

The following example of full matrix illustrates this corollary. For $A = [a_{ij}]_{n\times n}$, where

$$a_{ij} = \begin{cases} 2-n & \text{for } i = j, \\ 1 & \text{for } i \neq j, \end{cases}$$

we have $A^{-1} = [\widehat{a}_{ij}]_{n\times n}$, where

$$\widehat{a}_{ij} = \begin{cases} 0 & \text{for } i = j, \\ \dfrac{1}{n-1} & \text{for } i \neq j. \end{cases}$$

Answer to the problem fundamental for this paper, about the number of zeros in the inverses of full matrices, is given by the following theorem:

Theorem 5. *If $A \in GL(n, \mathbb{R})$ is a full matrix then A^{-1} contains at most $\frac{1}{2}(n-2)(n+3)$ zeros. Moreover, matrix A^{-1} contains in each row and in each column at most $n-2$ zeros.*

Proof. Proof of Theorem 5 runs in a similar way as proof of Theorem 3. □

Condition appearing in the above theorem (about containing by A^{-1} in each row and in each column at most $n-2$ zeros) is not a sufficient condition for A to be a full matrix. For example, we have

$$\begin{bmatrix} \mathbf{0}_{n\times n} & B \\ B & \mathbf{0}_{n\times n} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0}_{n\times n} & B^{-1} \\ B^{-1} & \mathbf{0}_{n\times n} \end{bmatrix},$$

where [13, problem 851]:

$$B = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}_{n\times n}$$

and

$$B^{-1} = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}_{n\times n}.$$

Next example describes the $n \times n$ matrices containing exactly $\frac{1}{2}(n-2)(n+3)$ zeros, for $n \geq 3$, and inverses of which are the full matrices.

Example 1. Let $G_n(k; x)$, where $k, n \in \mathbb{N}, x \in \mathbb{C},\ k \leq n-1$, denotes the $n \times n$ matrix, elements $g(i,j)$, $1 \leq i,\ j \leq n$, of which are defined in the following way

$$g(i,j) = \begin{cases} x & \text{for } (i,j) = (k+l, l),\quad 1 \leq l \leq n-k, \\ 1 & \text{for } (i,j) = (1, n-1),\ (1,n),\ (2,n),\ \text{and for all other cases when } i > j, \\ 0 & \text{for all other cases when } i \leq j. \end{cases}$$

For example, we have

$$G_4(1;x) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ x & 0 & 0 & 1 \\ 1 & x & 0 & 0 \\ 1 & 1 & x & 0 \end{bmatrix}, \qquad G_4(2;x) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \end{bmatrix},$$

$$G_4(3;x) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ x & 1 & 1 & 0 \end{bmatrix}.$$

The following facts are very interesting:

1° All inverse matrices $G_n^{-1}(k;x),\ k = 1, 2, 3,\ n \geq 3$, are full matrices for all $x \in \mathbb{C}$ which are sufficiently large with regard to their modulus. This follows from the fact, easily to prove, that all elements of matrices $\det(G_n(k;x))\, G_n^{-1}(k;x)$ are the nonzero polynomials of one variable x over $\mathbb{Z}$ (including also the polynomials of zero degree).

2° None matrix $G_n^{-1}(k;x),\ k \geq 4,\ n \geq 5$, is a full matrix for any $x \in \mathbb{C}$ since it contains certain elements equal to zero. For example, the third row of the matrix $G_n^{-1}(4;x),\ n \geq 5$, has the form

$$\begin{bmatrix} 0 & 0 & p_n(x) & -p_n(x) & \underbrace{0 \ldots 0}_{n-4 \text{ zeros}} \end{bmatrix},$$

where $p_n(x) \in \mathbb{Z}_n[x]$ and $\deg p_n(x) \geq 1$.

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References

[1] R. Wituła, N. Gawrońska, D. Słota, Basic properties of the full matrices, Zesz. Nauk. Pol. Śl. s. Mat. Stos. 1 (2011) 69–80.
[2] D. Słota, T. Trawiński, R. Wituła, Inversion of dynamic matrices of HDD head positioning system, Appl. Math. Modelling 35 (2011) 1497–1505.
[3] R. Wituła, D. Słota, M. Kampik, Some properties of full matrices, Appl. Math. Comput (submitted for publication).
[4] S.-Q. Shen, J.-M. Cen, Y. Hao, On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers, Appl. Math. Comput. 217 (2011) 9790–9797.
[5] F. Ding, Transformations between some special matrices, Comput. Math. Appl. 59 (2010) 2676–2695.
[6] F. Ding, T. Chen, On iterative solutions of general coupled matrix equations, SIAM J. Control Optim. 44 (2006) 2269–2284.
[7] F. Ding, T. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, Systems Control Lett. 54 (2005) 95–107.
[8] F. Ding, T. Chen, Gradient based iterative algorithms for solving a class of matrix equations, IEEE Trans. Automat. Control 50 (2005) 1216–1221.
[9] F. Ding, P.X. Liu, J. Ding, Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle, Appl. Math. Comput. 197 (2008) 41–50.
[10] L. Xie, J. Ding, F. Ding, Gradient based iterative solutions for general linear matrix equations, Comput. Math. Appl. 58 (2009) 1441–1448.
[11] J. Ding, Y.J. Liu, F. Ding, Iterative solutions to matrix equations of form $A_iXB_i = F_i$, Comput. Math. Appl. 59 (2010) 3500–3507.
[12] L. Xie, Y.J. Liu, H.Z. Yang, Gradient based and least squares based iterative algorithms for matrix equations $AXB + CX^TD = F$, Appl. Math. Comput. 217 (2010) 2191–2199.
[13] I.V. Proskuryakov, Problems in Linear Algebra, Mir, Moscow, 1978.